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B.Sc.–III

MATHEMATICS HONS.: Paper–V

group: B. (Multiple Integrals

Contents :→ Surface Integrals

Remark :→ Straight Line segment joining two points z_1 and z_2:–

Equation of the line segment joining z_1 and z_2 is

$$\boxed{z = t z_2 + (1-t) z_1} \quad \text{for } t \in [0,1]$$

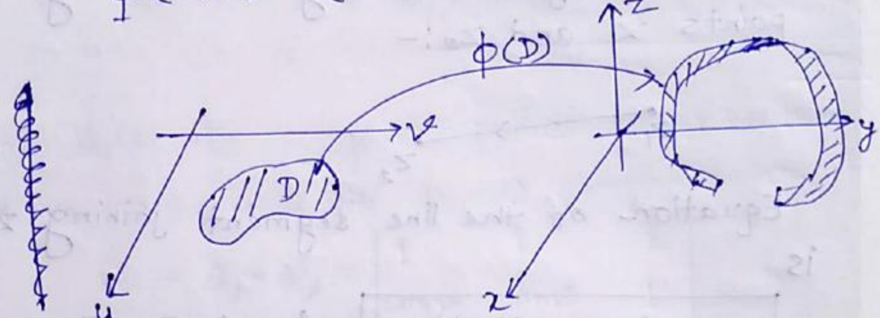
Surfaces :→ A surface S may be represented by $F(x,y,z)=0$.

The parametric representation of S is of the form $R(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$ and the continuous functions $u = \phi(t)$ and $v = \psi(t)$ of real parameter t represent a curve C on this surface S.

Remark :→ Parametrized Surface

A parametrized surface is a vector-valued function $\phi : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, where D is some domain in $\mathbb{R}^2$. The geometric surface S corresponding to the function ϕ is its image: $S = \phi(D)$. We write

$$\phi(u, v) = (x(u,v),\ y(u,v),\ z(u,v))$$

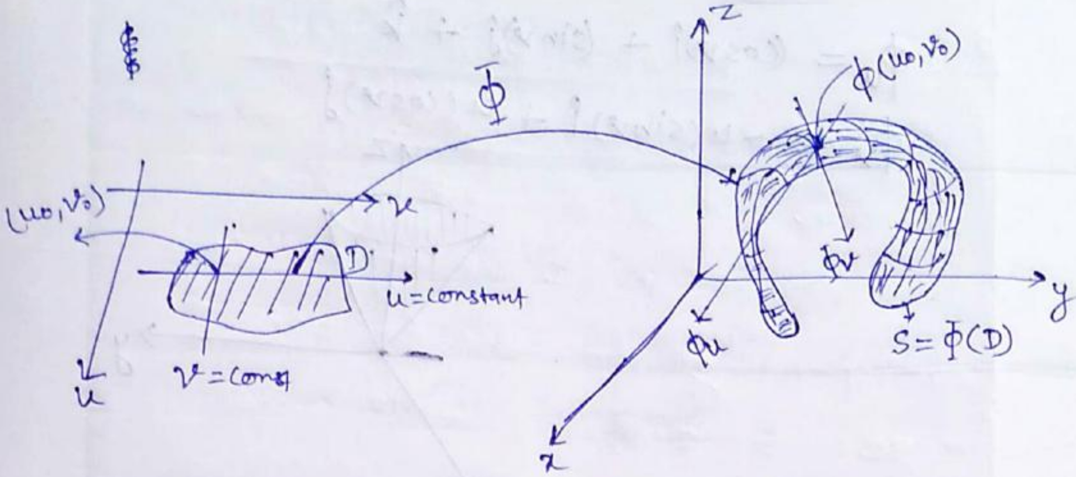


Fixing u at u_0, we consider the parametrized curve $c(v) = \phi(u_0, v)$, whose image is a curve on the surface. The Tangent vector to this curve is the derivative w.r.to v.

$$\phi_v = \frac{\partial \phi}{\partial v} = \frac{\partial x}{\partial v}(u_0, v_0)\,\hat{i} + \frac{\partial y}{\partial v}(u_0, v_0)\,\hat{j} + \frac{\partial z}{\partial v}(u_0, v_0)\,\hat{k}$$

similarly, if we fix v and consider the curve $b(u) = \phi(u, v_0)$, we obtain the tangent vector to this curve at $\phi(u_0, v_0)$, given by

$$\phi_u = \frac{\partial \phi}{\partial u} = \frac{\partial x}{\partial u}(u_0, v_0)\,\hat{i} + \frac{\partial y}{\partial u}(u_0, v_0)\,\hat{j} + \frac{\partial z}{\partial u}(u_0, v_0)\,\hat{k}$$



The tangent vectors ϕ_u and ϕ_v are tangent to curves on a surface S, and hence tangent to S.

$\because$ ϕ_u and ϕ_v are tangent to the surface, $\phi_u \times \phi_v$ is normal to it.

$$\therefore \hat{n} = \phi_u \times \phi_v = \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v}$$

<u>Example①</u> Let $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be given by

$$x = u\cos v,\ y = u\sin v,\ z = u, \text{ where } u \geq 0$$

Find the equation of ~~normal~~ tangent plane at $\phi(1,0)$.

<u>Solution:→</u>

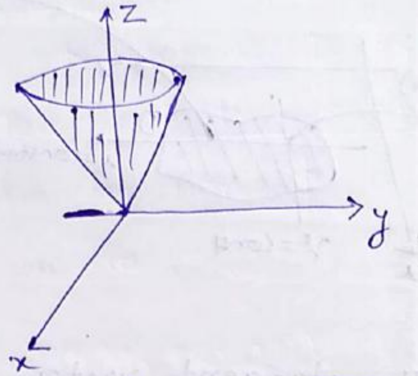
~~Note:→~~ $\cancel{\phi_u \times \phi_v = A\hat{i}+B\hat{j}+C\hat{k}}$

These equations describe the surface $z = \sqrt{x^2+y^2}$ and $z \geq 0$. This surface is a cone with a vertex at $(0,0,0)$.

We compute the partial derivatives:

$\phi_u = (\cos v)\hat{i} + (\sin v)\hat{j} + \hat{k}$

$\phi_v = -u(\sin v)\hat{i} + u(\cos v)\hat{j}$



The surface $z = \sqrt{x^2+y^2}$ is a cone.

And thus the tangent plane $\phi(u,v)$ is the set of vectors through $\phi(u,v)$ perpendicular to

$$\hat{n} = \phi_u \times \phi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 1 \\ -u \cdot \sin v & u\cos v & 0 \end{vmatrix}$$

$$= -u\cos v\,\hat{i} - u\sin v\,\hat{j} + u\hat{k}$$

$\therefore \hat{n} = (-u\cos v, -u\sin v, u)$

If this vector is non-zero. since $\phi_u \times \phi_v$ is equal to 0 when $u=0$, there is no tangent plane at $\phi(0,v) = (0,0,0)$. However we can find an equation of the tangent plane at all the points where $\phi_u \times \phi_v \neq 0$.

At the point $\phi(1,0) = (1,0,1)$

$\hat{n} = \phi_u \times \phi_v = (-1,0,1) = -\hat{i} + \hat{k}$

$\therefore$ we have the vector $\hat{n}$ normal to the surface and a point $(1,0,1)$ on the surface.

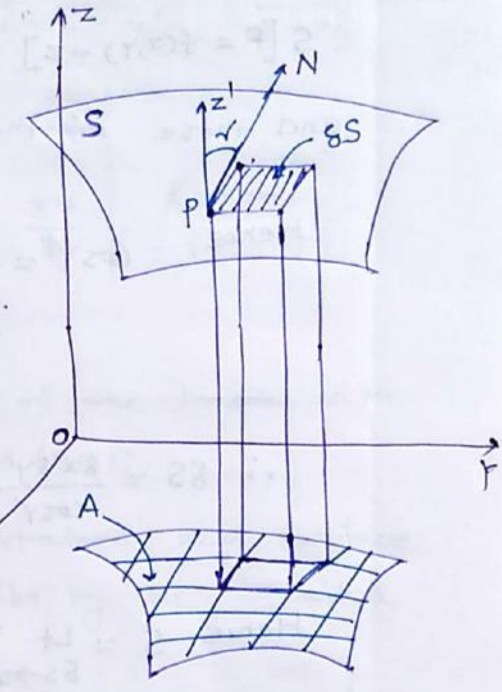
$\therefore$ Eq^n of tangent is $A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$

Where $\hat{n} = A\hat{i} + B\hat{j} + C\hat{k}$

$-(x-1) + (z-1) = 0$

$\Rightarrow x = z$ Ans.

Remark :→ Area of a curved surface.



Consider a point P of the surface $S : z = f(x,y)$.

Let its projection on the xy-plane be the region A. Divide it into area elements by drawing lines parallel to the axes X & Y.

On the element $\delta x \cdot \delta y$ as base, erect a cylinder having generators parallel to OZ & meeting the surface S in an element of area δS.

As $\delta x \delta y$ is the projection of δS on the xy-plane.

$\therefore$ $\delta x \delta y = \delta S \cos\gamma$, where γ is the angle between xy-plane and the tangent plane to S at P i,e; it is the angle between the Z-axis and the normal to S at P.

Now since the direction cosines of the normal to the surface $F(x,y,z) = 0$ are parallel to $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$.

$\therefore$ The direction cosines of the normal to S $[F = f(x,y) - z]$ are proportional to $-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1$.

and those of the z-axis are 0, 0, 1.

Hence, $$\cos\gamma = \frac{1}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}$$

$$\therefore \delta S = \frac{\delta x\, \delta y}{\cos\gamma} = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\cdot \delta x . \delta y.$$

Hence $$S = \underset{\delta S \to 0}{Lt} \sum \delta S = \iint_A \left(\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\right) dx . dy.$$

<u>Remark</u> : →

For a geometric surface S that is the image of the parametrization ϕ, we ~~have~~ write $\phi_u = \frac{\partial \phi}{\partial u}$, $\phi_v = \frac{\partial \phi}{\partial v}$. Then

$$d\vec{S} = (\phi_u \times \phi_v)\, du\, dv$$

and
$$dS = \| d\vec{S} \| = \| \phi_u \times \phi_v \|\, du\, dv$$

So that $$\hat{n} = \frac{d\vec{S}}{dS} \quad \& \quad d\vec{S} = \hat{n}\, dS$$

Where $\hat{n}$ is ~~the~~ a unit normal vector to the surface. The <u>surface area</u> is

$$A = \iint_D dS = \iint_D \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2} \cdot du\,dv$$

Where $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ & so on.

Example(2):- Find the area of the tringle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

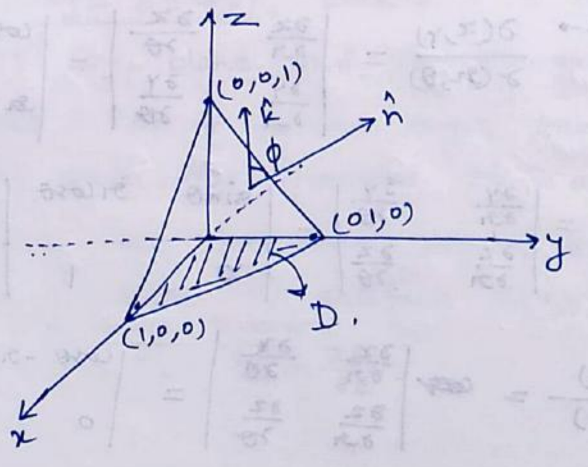
Solution:→ The tringle is contained in a surface, namely, the plane describe by the equation $x+y+z=1$.

Since the surface is a plane, the angle ϕ is constant & a unit normal vector is

$\hat{n} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

Thus $\cos\phi = \hat{n}\cdot\hat{k} = \frac{1}{\sqrt{3}}$ & so area is

$$A = \iint_D dS = \iint_D \frac{dx\,dy}{1/\sqrt{3}} = \sqrt{3}\ (\text{area}D) = \frac{\sqrt{3}}{2}.$$



Integral of a scalar Function over a surface

Let S be a surface parametrized by a mapping $\phi: D \longrightarrow S \subset \mathbb{R}^3$, $\phi(u,v) = (x(u,v), y(u,v), z(u,v))$.

If $f(x,y,z)$ is a real-valued continuous function defined on S, we define the integral of f over S to be

$$\iint_S f(x,y,z)\, dS = \iint_S f\, dS = \iint_D f(\phi(u,v)) \|\phi_u \times \phi_v\|\, du\, dv$$

i,e,

$$\iint_S f\, dS = \iint_D f(x(u,v), y(u,v), z(u,v)) \times \sqrt{\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2} \times du\, dv$$

Example:→ consider the helicoid $x = r\cos\theta$, $y = r\sin\theta$, $z = \theta$, where $0 \le \theta \le 2\pi$ & $0 \le r \le 1$. Let f be given by $f(x,y,z) = \sqrt{x^2+y^2+1}$. Find $\iint_S f\, dS$.

Solution:→ $$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\frac{\partial(y,z)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\theta & r\cos\theta \\ 0 & 1 \end{vmatrix} = \sin\theta$$

$$\frac{\partial(x,z)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ 0 & 1 \end{vmatrix} = \cos\theta$$

Also, $f(r\cos\theta, r\sin\theta, \theta) = \sqrt{1+r^2}$. Therefor

$$\iint_D f(x,y,z)\, dS = \iint_D f(\phi(r,\theta)) \, \| \phi_r \times \phi_\theta \| \, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \sqrt{1+r^2}\,\sqrt{1+r^2}\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[\frac{r^3}{3} + r \right]_0^1 d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{4}{3}\, d\theta = \frac{4}{3} \times [\theta]_0^{2\pi} = \frac{8}{3}\pi \text{ Ans.}$$

Surface Integrals :→

The surface integral of a vector field $\vec{F}$ on $\mathbb{R}^3$ over a parametrized surface $\phi : D \longrightarrow \mathbb{R}^3$ is the number

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\phi(u,v)) \cdot (\phi_u \times \phi_v)\, du\, dv.$$

$\because d\vec{S} = \hat{n}\, dS$, where $\hat{n} = \dfrac{\phi_u \times \phi_v}{\|\phi_u \times \phi_v\|}$ is a unit normal, we can write the surface integral as

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{n}\, dS$$

Example :→ Let D be the rectangle in the ~~θφ~~ $\theta\phi$-plane defined by

$$0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi,$$

and let the surface S be defined by the parametrization $\Phi : D \longrightarrow \mathbb{R}^3$ given by

$$x = \cos\theta \sin\phi, \quad y = \sin\theta \sin\phi, \quad z = \cos\phi$$

Here θ & ϕ are the angles of spherical coordinates, and S is the unit sphere perametrized by ϕ. Let $\vec{r}$ be the position vector $\vec{r}(x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}$. Compute $\iint_S \vec{r} \cdot d\bar{S}$.

Solution :→

$$\because \Phi = \cos\theta \sin\phi \, \hat{i} + \sin\theta \sin\phi \, \hat{j} + \cos\phi \, \hat{k}$$

$$\therefore \Phi_\theta = (-\sin\phi \sin\theta)\hat{i} + (\sin\phi \cos\theta)\hat{j}$$

$$\Phi_\phi = (\cos\theta \cos\phi)\hat{i} + (\sin\theta \cos\phi)\hat{j} - (\sin\phi)\hat{k}$$

& hence

$$\Phi_\theta \times \Phi_\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\phi \sin\theta & \sin\phi \cos\theta & 0 \\ \cos\theta \cos\phi & \sin\theta \cos\phi & -\sin\phi \end{vmatrix}$$

$$\Phi_\theta \times \Phi_\phi = (-\sin^2\phi \cos\theta)\hat{i} - (\sin^2\phi \sin\phi)\hat{j} - (\sin\phi \cos\phi)\hat{k}$$

Then

$$\vec{r} \cdot (\Phi_\theta \times \Phi_\phi) = (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\Phi_\theta \times \Phi_\phi)$$

$$= [(\cos\theta \sin\phi)\hat{i} + (\sin\theta \sin\phi)\hat{j} + (\cos\phi)\hat{k}]$$

$$\cdot (-\sin\phi)[\sin\phi \cos\theta \, \hat{i} + (\sin\phi \sin\theta)\hat{j} + (\cos\phi)\hat{k}]$$

$$= (-\sin\phi)(\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi)$$

$$= -\sin\phi$$

Thus,

$$\iint_S \vec{r}\cdot d\vec{S} = \iint_D \vec{r}\cdot(\Phi_\theta \times \Phi_\phi)\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} -\sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi}(-2)\, d\theta = -4\pi \quad \underline{\underline{\text{Ans}}}.$$